

A new construction for the shortest non-trivial element in the lower central series

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ABSTRACT. We provide a new upper bound for the length for the shortest non-trivial element in the lower central series $\gamma_n(\mathbb{F}_2)$ of the free group on two generators. We prove that it has an asymptotic behaviour of the form $O(n^{\log_\varphi(2)})$, where $\varphi = 1.618\dots$ is the golden ratio. This new technique is used to provide new estimates on the length of laws for finite groups and on almost laws for compact groups.

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1. Introduction

This is a companion paper to [7] and we will refer to it for most of the definitions and some of the results. It has been conjectured by Malestein and Putman [5] that the length of the shortest non-trivial element in the derived series and the lower central series in the free group of rank 2 has an asymptotic behaviour of the form $O(n^2)$. The author and A. Thom [7] disproved this conjecture by some construction which provides a better upper bound. In this note we slightly improve this upper bound. The new construction suggests that this asymptotic behaviour is of the form $O(n^{\log_\varphi(2)})$, where φ is the golden ratio. The most interesting thing in this new construction is that it is enough to multiply the word length by 2 to increase the depth of the central series by some Fibonacci number. Indeed, the lower central series steps are $\gamma_{f_{i+2}}(\mathbb{F}_2)$ where $(f_n)_{n \in \mathbb{N}}$ is the sequence of Fibonacci numbers ($f_0 := 0$, $f_1 := 1$, and $f_{n+2} := f_{n+1} + f_n$ for $n \in \mathbb{N}$), see Remark 2.9. In this paper we conjecture this asymptotic behaviour $O(n^{\log_\varphi(2)})$ to be sharp and it is still unclear how to prove this conjecture. However, we give some remarks on the length of some elements in this construction which may support

this conjecture, see Remark 2.10 and Remark 2.11. Moreover, we also improve some previous results for the length of laws for finite groups, see Corollary 2.7 and Corollary 2.8 and on almost laws for compact groups, see Section 3.1. We write $f(n) \preceq g(n)$ if there is a constant C such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. The author and A. Thom [7] studied the growth of the function $\alpha(n) := \min\{\ell(w) \mid w \in \gamma_n(\mathbb{F}_2) \setminus \{e\}\}$ where $\ell: \mathbb{F}_2 \rightarrow \mathbb{N}$ is length function with respect to the generating set $\{a, a^{-1}, b, b^{-1}\}$. We can think of $\alpha(n)$ as the girth of the Cayley graph of the group $\mathbb{F}_2/\gamma_n(\mathbb{F}_2)$ with respect to the image of the natural generating set of \mathbb{F}_2 . In view of [7, Lemma 2.1] we set $\alpha := \lim_{n \rightarrow \infty} \frac{\log_2(\alpha(n))}{\log_2(n)}$. The author and A. Thom [7] provides upper bound for α , that is $\alpha \leq \frac{\log_2(3+\sqrt{17})-1}{\log_2(1+\sqrt{2})} = 1,4411\dots$. For more background on free groups you can see [3] and [4].

We provide a new construction of words in the lower central series which slightly improves the upper bound of α in Theorem [7]. Our main result is the following:

THEOREM 1.1. *Let \mathbb{F}_2 be the free group on two generators and $(\alpha(n))_{n \in \mathbb{N}}$, and α be defined as above. We have*

$$\alpha \leq \log_\varphi(2) = 1,440\dots$$

or equivalently $\alpha(n) \preceq n^{\log_\varphi(2)+\varepsilon}$ for all $\varepsilon > 0$.

We state this conjecture:

CONJECTURE 1.2. $\alpha = \log_\varphi(2)$.

2. The new construction

LEMMA 2.1. *For $w_1, w_2 \in \mathbb{F}_2$, and $n \in \mathbb{Z}$, we have*

$$(1) \quad [w_1, w_2] = [w_1 w_2^n, w_2] \quad \text{and} \quad [w_1, w_2] = [w_1, w_2 w_1^n].$$

PROOF. We compute:

$$[w_1 w_2^n, w_2] = w_1 w_2^n w_2 w_2^{-n} w_1^{-1} w_2^{-1} = w_1 w_2 w_1^{-1} w_2^{-1} = [w_1, w_2].$$

In the same way we show that $[w_1, w_2] = [w_1, w_2 w_1^n]$ and this proves the claim. \square

We set $a_0 := b^{-1}$, $b_0 := aba^{-1}$ and define recursively

$$a_n := a_{n-1} b_{n-1}, \quad b_n := a_{n-1}^{-1} b_{n-1}^{-1}, \quad \text{for all } n \in \mathbb{N}.$$

LEMMA 2.2 (length of a_n and b_n). *Let $n > 0$. Then a_n and b_n have the following reduced representations:*

$$a_n = \begin{cases} b^{-1} \dots b^{-1} & : n \equiv 0 \pmod{3} \\ b^{-1} \dots ba^{-1} & : n \equiv 1 \pmod{3} \\ b^{-1} \dots b^{-1} a^{-1} & : n \equiv 2 \pmod{3} \end{cases} \quad \text{and} \quad b_n = \begin{cases} ab \dots ba^{-1} & : n \equiv 0 \pmod{3} \\ b \dots b^{-1} a^{-1} & : n \equiv 1 \pmod{3} \\ ab^{-1} \dots b^{-1} & : n \equiv 2 \pmod{3} \end{cases}.$$

Thus, there is no cancellation in the product $a_n^{-1} b_n^{-1}$ for all $n \in \mathbb{N}$. The product $a_n b_n$ involves cancellation if and only if $n \equiv 2 \pmod{3}$, where the term $a^{-1}a$ cancels out. Hence, for $n \in \mathbb{N}$

we have

$$\ell(a_{n+1}) = \ell(a_n) + \ell(b_n) - \begin{cases} 2 & : n \equiv 2 \pmod{3} \\ 0 & : \text{otherwise} \end{cases} \quad \text{and} \quad \ell(b_{n+1}) = \ell(a_n) + \ell(b_n).$$

Solving the recursion yields

$$\ell(a_n) = \begin{cases} \frac{13 \cdot 2^n - 6}{7} & : n \equiv 0 \pmod{3} \\ \frac{13 \cdot 2^n + 2}{7} & : n \equiv 1 \pmod{3} \\ \frac{13 \cdot 2^n + 4}{7} & : n \equiv 2 \pmod{3} \end{cases} \quad \text{and} \quad \ell(b_n) = \begin{cases} \frac{13 \cdot 2^n + 8}{7} & : n \equiv 0 \pmod{3} \\ \frac{13 \cdot 2^n + 2}{7} & : n \equiv 1 \pmod{3} \\ \frac{13 \cdot 2^n + 4}{7} & : n \equiv 2 \pmod{3} \end{cases}.$$

In particular, there exists a constant $C' > 0$, such that $\ell(a_n) \leq C' \cdot 2^n$ for all n .

PROOF. At first we mention why it is enough to show the statement about the reduced representations for a_n and b_n (for $n > 0$).

Namely, from these formulas it follows directly that there is no cancellation in the product $a_n^{-1}b_n^{-1}$ and the described cancellation in a_nb_n (for $n = 0$ one checks this explicitly). From this we deduce the stated recursion formulas for $\ell(a_n)$ and $\ell(b_n)$. A straightforward induction shows the correctness of the given explicit formulas for $\ell(a_n)$ and $\ell(b_n)$.

Finally, let us prove the statement about the reduced representations of a_n and b_n by induction on $n \in \mathbb{N}_{>0}$. For $n = 1$ we have $a_1 = b^{-1}aba^{-1}$ and $b_1 = bab^{-1}a^{-1}$, so the claim is true. Now let us do one step of the induction, e.g. the case $n \equiv 0 \pmod{3}$. Then $a_{n+1} = b^{-1} \dots b^{-1}ab \dots ba^{-1} = b^{-1} \dots ba^{-1}$ and $b_{n+1} = b \dots bab^{-1} \dots b^{-1}a^{-1} = b \dots b^{-1}a^{-1}$ as claimed, since $n + 1 \equiv 1 \pmod{3}$. The cases $n \equiv 1, 2 \pmod{3}$ are treated analogously. \square

We set

$$(2) \quad \gamma(w) := \max\{n \mid w \in \gamma_n(\mathbb{F}_2)\}. \quad \forall w \in \mathbb{F}_2.$$

Clearly,

$$(3) \quad \gamma([w_1, w_2]) \geq \gamma(w_1) + \gamma(w_2),$$

and

$$(4) \quad \alpha(\gamma(a_n)) \leq \ell(a_n) \text{ for all } n \in \mathbb{N}.$$

LEMMA 2.3. Let $\sigma, \tau \in \text{Aut}(\mathbb{F}_2)$ be defined by $\sigma : a \mapsto a^{-1}, b \mapsto b^{-1}$ and $\tau : a \mapsto a, b \mapsto b^{-1}$. Let $n \in \mathbb{N}$.

- (1) If $n \equiv 0 \pmod{3}$ then $a\sigma(a_n)a^{-1} = b_n$ (and so $a_n = \sigma^{-1}(a^{-1}b_na) = \sigma(a^{-1}b_na) = a\sigma(b_n)a^{-1}$).
- (2) If $n \equiv 1 \pmod{3}$ then $\tau(a_n) = b_n$ (and so $\tau(b_n) = \tau^2(a_n) = a_n$).
- (3) If $n \equiv 2 \pmod{3}$ then $\tau(a_n) = b_n^{-1}$.

PROOF. At first we prove statements (1) and (2) by induction on n . For $n = 0$ we have $a_0 = b^{-1} \xrightarrow{a\sigma(\cdot)a^{-1}} aba^{-1} = b_0$, and for $n = 1$ we have $a_1 = b^{-1}aba^{-1} \xrightarrow{\tau} bab^{-1}a^{-1}$ verifying the

claim. Now, let $\eta = a\sigma(\cdot)a^{-1}$ if $n \equiv 0 \pmod{3}$ and $\eta = \tau$ if $n \equiv 1 \pmod{3}$. By definition it holds for $n \in \mathbb{N}$ that

$$a_{n+3} = [a_n b_n, a_n^{-1} b_n^{-1}], \quad \text{and} \quad b_{n+3} = [b_n a_n, b_n^{-1} a_n^{-1}].$$

Thus, the induction hypothesis gives us $b_{n+3} = [\eta(a_n)\eta(b_n), \eta(a_n)^{-1}\eta(b_n)^{-1}] = \eta([a_n b_n, a_n^{-1} b_n^{-1}]) = \eta(a_{n+3})$.

At last we show statement (3). Let $n \equiv 2 \pmod{3}$. Then, by statement (2) and by definition we obtain $b_n^{-1} = b_{n-1} a_{n-1} = \tau(a_{n-1})\tau(b_{n-2}) = \tau(a_{n-1} b_{n-1}) = \tau(a_n)$. \square

An immediate corollary of the last lemma is

COROLLARY 2.4. *We have $\gamma(a_n) = \gamma(b_n)$.*

PROOF. By Lemma 2.3 we get for a characteristic subgroup $G \subseteq \mathbb{F}_2$ that $a_n \in G$ if and only if $b_n \in G$. \square

In the following lemma, we estimate $\gamma(a_n)$.

LEMMA 2.5. *We have $\gamma(a_{n+2}) \geq \gamma(a_{n+1}) + \gamma(a_n)$ for all $n \in \mathbb{N}$. In particular, there exists a constant $C > 0$, such that $\gamma(a_n) \geq C \cdot \varphi^n$, where $\varphi = 1.618\dots$ is the golden ratio.*

PROOF. We compute $a_{n+2} = [a_n, b_n] \stackrel{(1)}{=} [a_n b_n, b_n]$. Then it follows from 3 and 2.4 that $\gamma(a_n) \geq \gamma(a_{n-1}) + \gamma(a_{n-2})$. The estimate on $\gamma(a_n)$ follows from the fact that the golden ratio φ is the largest root of the polynomial $p(t) = t^2 - t - 1$. \square

Now we are ready to prove Theorem 1.1, the proof follows from the following proposition.

PROPOSITION 2.6. *We have $\alpha(n) \leq C'' \cdot n^{\log_\varphi(2)}$, $C'' \in \mathbb{R}_{>0}$ for infinitely many $n \in \mathbb{N}$ and thus $\alpha \leq \log_\varphi(2)$.*

PROOF. By Lemma 2.5, we get $n \leq \frac{\log_2(\gamma(a_n)) - \log_2(C)}{\log_2(\varphi)}$ and hence

$$\begin{aligned} \alpha(\gamma(a_n)) &\stackrel{(4)}{\leq} \ell(a_n) \\ &\leq C' \cdot 2^n \\ &\leq C' \exp\left(\frac{\log(2) \cdot (\log_2(\gamma(a_n)) - \log_2(C))}{\log_2(\varphi)}\right) \\ &= C' \exp\left(\frac{-\log(2) \log_2(C)}{\log_2(\varphi)}\right) \cdot (\gamma(a_n))^{\log_\varphi(2)}. \end{aligned}$$

This proves the claim. \square

For any group G and a word $w \in \mathbb{F}_2$, the word map $w : G \times G \rightarrow G$ is a natural map which is given by evaluation. We say that w is a law for G if the image of the corresponding word map is the identity element, i.e., $w(g, h) = 1$ for all $g, h \in G$. By the method of Khalid Bou-Rabee [2] and [7, Theorem 2.1] A Thom [8] proved that for $n \in \mathbb{N}$ there exists a word $w_n \in \mathbb{F}_2$ of length bounded by $O(\log(n)^{1.4411})$ which is a law for all nilpotent group of size at most n . Using the new upper bound of α in Theorem 1.1, we can improve this little bit:

COROLLARY 2.7. *Let $n \in \mathbb{N}$. There exists $w_n \in \mathbb{F}_2$ of length bound by $O(\log(n)^{\log_\varphi(2)})$ which is a law for all nilpotent group of size at most n .*

Moreover, following the method of A. Thom [8], we can improve his result [8, Proposition 3.2] on solvable groups little bit:

COROLLARY 2.8. *Let $n \in \mathbb{N}$. There exists $w_n \in \mathbb{F}_2$ of length bound by $O((\log(n))^{(\log_\varphi(2)+2.890457)})$ which is a law for all solvable groups of size at most n .*

REMARK 2.9. We have $a_0 = b^{-1} \in \gamma_1(\mathbb{F}_2)$, $a_1 = [b^{-1}, a] \in \gamma_2(\mathbb{F}_2)$, $a_2 = [b^{-1}, aba^{-1}] = [b^{-1}, a][b, a] = [[b^{-1}, a], b] \in \gamma_3(\mathbb{F}_2)$, and $a_3 = [a_1, b_1] = [[b^{-1}, a], [b, a]] \stackrel{(1)}{=} [[b^{-1}, a][b, a], [b, a]] = [[b^{-1}, a], b], [b, a]] \in \gamma_5(\mathbb{F}_2)$. So according to the construction $\gamma(a_n) \geq \gamma(a_{n-1}) + \gamma(a_{n-2})$ you can easily see that $a_n \in \gamma_{f_{n+2}}(\mathbb{F}_2)$, where f_m is the m -th Fibonacci number and is given by formula $f_m = \frac{\varphi^m - (-\varphi)^m}{\sqrt{5}}$, $m \in \mathbb{N}$.

In the following two remarks we show the length of the shortest element in some central series of this new construction which suggests the conjecture 1.2.

REMARK 2.10. We have:

- (1) $\min\{\ell(w) : w \in \gamma_1(\mathbb{F}_2) \setminus \{1\}\} = 1$.
- (2) $\min\{\ell(w) : w \in \gamma_2(\mathbb{F}_2) \setminus \{1\}\} = 4$. Indeed $w = a^{n_1}b^{m_1} \dots a^{n_k}b^{m_k} \in \gamma_2(\mathbb{F}_2)$ if and only if $\sum_i n_i = \sum_i m_i = 0$. So, the shortest non-trivial element that satisfies this condition is of length 4.
- (3) $\min\{\ell(w) : w \in \gamma_3(\mathbb{F}_2) \setminus \{1\}\} = 8$. Indeed if we consider any element of length 6 in $\gamma_2(\mathbb{F}_2)$ (which are finitely many elements) $[b^{-1}, a][a, b] = [b^{-1}, a][b, a][a, b]^2 = [[b^{-1}, a], b][a, b]^2 \in \gamma_2(\mathbb{F}_2) \setminus \gamma_3(\mathbb{F}_2)$ since $[a, b]^2 \in \gamma_2(\mathbb{F}_2) \setminus \gamma_3(\mathbb{F}_2)$. Also if we consider the quotient $\gamma_2(\mathbb{F}_2)/\gamma_3(\mathbb{F}_2)$ (which is free abelian group of finite rank, see from example [9], and [4]), then one can easily see that $[a, b]^2$ is the only factor of $[[b^{-1}, a], b][a, b]^2$ that survives in this quotient, hence $[[b^{-1}, a], b][a, b]^2 \notin \gamma_3(\mathbb{F}_2)$.

REMARK 2.11. It follows from Lemma 2.1 that the word $a_4 = [[b^{-1}, a][b, a], [a, b^{-1}][a, b]] \in \gamma_8(\mathbb{F}_2)$, $\ell(a_4) = 30$ and this suggests asymptotic behaviour of the form $O(n^\nu)$, where $\nu = 0.6113\dots$ For the word $w = [[b^{-1}, a][a, b], [a, b^{-1}][b, a]]$, we have that $\ell(w) = 28 < \ell(a_4) = 30$ but it follows from Lemma 2.1 and Remark 2.10, (3) that $w \in \gamma_7(\mathbb{F}_2)$ since $\ell([b^{-1}, a][a, b]) = 6$ (so $[b^{-1}, a][a, b] \in \gamma_2(\mathbb{F}_2) \setminus \gamma_3(\mathbb{F}_2)$) and $[a, b^{-1}][b, a] = [a, b]([b, a][a, b^{-1}])[b, a] \in \gamma_2(\mathbb{F}_2) \setminus \gamma_3(\mathbb{F}_2)$. Thus the word $w = [[b^{-1}, a][a, b], [a, b^{-1}][b, a]]$ suggests the asymptotic behaviour of the form $O(n^\mu)$, where $\mu = 0.583\dots < \nu$.

REMARK 2.12. We can also consider this construction, we set $a'_0 = a$, $b'_0 = b$, and define recursively:

$$a'_n = a'_{n-1}b'_{n-1}, \quad b'_n = a'^{-1}_{n-1}b'^{-1}_{n-1}.$$

You can easily see that the products here involve no cancellations and $\ell(a'_n) = 2\ell(a'_{n-1})$ and also $\gamma(a'_n) \geq \gamma(a'_{n-1}) + \gamma(a'_{n-2})$ hence this construction suggests the asymptotic behaviour of the form $O(n^{\log_\varphi(2)})$.

REMARK 2.13. For a subgroup $\Gamma \subset \mathbb{F}_2$, we define $\text{girth}(\Gamma) := \min\{\ell(w) | w \in \Gamma \setminus \{e\}\}$. The Author and A. Thom [7] proved that for $\Gamma \subset \mathbb{F}_2$ is a normal subgroup. Then the following holds: $\text{girth}([\Gamma, \Gamma]) \geq 3 \cdot \text{girth}(\Gamma)$. We have $a_3 = [[b^{-1}, a], b], [a, b]]$, $\ell(w_1) = 14$, then $\text{girth}([\gamma_3(\mathbb{F}_2), \gamma_2(\mathbb{F}_2)]) \leq 14$. It follows from Remark 2.10 that $\text{girth}(\gamma_3(\mathbb{F}_2)) = 8$, and $\text{girth}(\gamma_2(\mathbb{F}_2)) = 4$, then we can easily see for two different normal subgroups that

$$\text{girth}([\gamma_3(\mathbb{F}_2), \gamma_2(\mathbb{F}_2)]) \leq 14 < 2 \cdot \text{girth}(\gamma_3(\mathbb{F}_2))$$

and

$$\text{girth}([\gamma_3(\mathbb{F}_2), \gamma_2(\mathbb{F}_2)]) \leq 14 > 3 \cdot \text{girth}(\gamma_2(\mathbb{F}_2)).$$

3. Almost laws for compact groups

Consider the word map on $\text{SU}(k)$ for $w \in \mathbb{F}_2$ where $\text{SU}(k)$ is the special unitary group. A. Thom [6] proved that there exists a sequence of nontrivial elements (w_n) in \mathbb{F}_2 , such that for every neighborhood $U \subset \text{SU}(k)$ of identity, there exists $N \in \mathbb{N}$ such that $w_n(\text{SU}(k) \times \text{SU}(k)) \subset U$ for all $n \geq N$. This sequence of words is called almost law for compact groups see, [1] for more details. For $u, v \in \text{SU}(k)$, there is a metric $d(u, v) := \|u - v\|$ where $\|\cdot\|$ is the operator norm. We set $L_k(w) := \max\{d(1_k, w(u, v)) \mid u, v \in \text{SU}(k)\}$. The author and A. Thom [7] proved that there exists an almost law (w_n) for $\text{SU}(k)$ such that there exists a constant $C > 0$ depending on k such that $L_k(w_n) \leq \exp(-C \cdot \ell(w_n)^\delta)$ with $\delta = \frac{\log_2(1+\sqrt{2})}{\log_2(3+\sqrt{17})-1} = 0.69391\dots$. Using this new construction we can improve this little bit.

THEOREM 3.1. *Let $k \in \mathbb{N}$. There exists an almost law $(w_n)_n$ for $\text{SU}(k)$ such that the following holds there exists a constant $C > 0$ such that*

$$L_k(w_n) \leq \exp\left(-C \cdot \ell(w_n)^{\log_2(\varphi)}\right)$$

where φ is the golden ratio.

PROOF. It follows from [6, Lemma 2.1], that

$$(5) \quad \|1 - u_1 u_2\| \leq 2\|1 - u_1\|\|1 - u_2\|, \text{ for } u_1, u_2 \in \text{SU}(k).$$

By [6, Corollary 3.3.] there exist words $w, v \in \mathbb{F}_2$ which generate a free subgroup and satisfy $L_k(w), L_k(v) \leq \frac{1}{3}$. Let us set $w_n := a_n(w, v)$. It follows from Lemma 2.1 that

$$(6) \quad \ell(w_n) \leq C'' \cdot 2^n$$

for some constant $C'' > 0$. On the other side, Equation (5) and the equation $a_{n+2} = [a_{n+1}, a_{n-1}]$ show that $L_k(w_n) \leq 2 \cdot L_k(w_{n-1})L_k(w_{n-2})$ or equivalently

$$-\log(2L_k(w_n)) \geq -\log(2L_k(w_{n-1})) - \log(2L_k(w_{n-2})).$$

Thus there exists a constant $D > 0$ (as in the proof of Lemma 2.5) such that

$$(7) \quad -\log(2L_k(w_n)) \geq D \cdot \varphi^n,$$

for some constant $D > 0$. Hence,

$$L_k(w_n) \stackrel{(7)}{\leq} \frac{1}{2} \exp(-D \cdot \varphi^n) \stackrel{(6)}{\leq} \exp\left(-C \cdot \ell(w_n)^{\log_2(\varphi)}\right)$$

for some constant C . This proves the claim. \square

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